# *On Three-Spreads Satisfying Four or More Homogeneous Linear Partial Differential Equations of the Second Order.*

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## INTRODUCTION.

1. In an article in the *Atti della Accademia Reale delle Scienze di Torino*,* SEGRE discussed the two-spreads which satisfy one or more homogeneous linear partial differential equations of the second order. The discussion here given follows the same order of ideas.

The equations of the three-spread are supposed to be given in the parametric form:

$$x_i = f_i(u_1, u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

The three-spread will be said to satisfy an homogeneous partial differential equation when each of the $n+1$ functions $f_i$ satisfies the equation.

2. It will first be determined under what conditions a three-spread may satisfy more than four homogeneous linear partial differential equations of the second order. It will next be shown that, if the three-spread satisfies four such equations, it has, at an arbitrary point, four tangents having contact of the second order with the three-spread. It will then be determined under what conditions two or more of these three-point tangents, at an arbitrary point, may be consecutive.

The functions $f_i$ are, in general, supposed to be analytic, although, in a large part of the work, this is a greater restriction than is necessary.

Differentiation will be denoted by indices, for example:

$$\frac{\partial f}{\partial u_1}, \quad \frac{\partial^2 f}{\partial u_3^2}, \quad \frac{\partial^3 f}{\partial u_1 \partial u_2 \partial u_3}$$

will be denoted, respectively, by $f^1$, $f^{33}$ and $f^{123}$.

* "Su una Classe di Superficie degl' Iperspazii Legate colle Equazioni Lineari alle Derivate Parziali di 2° Ordine." Vol. XLII (1907), p. 559.

3. Let $f_0 \neq 0$ in the region under consideration and let

$$g_i = \frac{f_i}{f_0}, \qquad i = 1, 2, \ldots, n.$$

Then the equations of the three-spread may be written non-homogeneously in the form:

$$x_i = g_i(u_1, u_2, u_3), \qquad i = 1, 2, \ldots, n.$$

Since the entity under consideration is, by hypothesis, a three-spread, the matrix,

$$\left\| \frac{\partial g_i}{\partial u_j} \right\|, \qquad \begin{matrix} i = 1, 2, \ldots, n, \\ j = 1, 2, 3, \end{matrix} \tag{1}$$

must be of rank three.

4. It follows that the functions $f_i$ do not all satisfy an homogeneous linear partial differential equation of the first order:

$$a_1 f^1 + a_2 f^2 + a_3 f^3 + af = 0.$$

For, suppose that such an equation were satisfied. It would then follow that the functions $g_i = \dfrac{f_i}{f_0}$ all satisfy

$$a_1 g^1 + a_2 g^2 + a_3 g^3 = 0.$$

Hence, since the matrix (1) is of rank three, it follows that $a_1 = a_2 = a_3 = 0$. Since $f_0 \neq 0$, it now follows that $a = 0$.

## On Three-Spreads Satisfying More Than Four Homogeneous Linear Partial Differential Equations of the Second Order.

5. No three-spread can satisfy more than six independent homogeneous linear partial differential equations of the second order. For, if it could satisfy seven such equations, it would have to satisfy the equation of the first order, which could be deduced from them. This, we have seen, is impossible.

6. If the three-spread satisfies six such equations, these equations must be reducible to the form

$$f^{hk} = a_{hk} f + b_{hk} f^1 + c_{hk} f^2 + d_{hk} f^3, \qquad h = 1, 2, 3; \; k = 1, 2, 3, \tag{1}$$

since, otherwise, from the six equations could be deduced one of the first order.

A three-spread which satisfies six equations of the above form lies in an $S_3$.*

* The notation $S_r$ will be used to denote a space of $r$ dimensions.

For, let $f_0 \neq 0$ in the region considered and let $g_i = \dfrac{f_i}{f_0}$. From the above six equations we obtain:

$$g^{hk} = \beta_{hk} g^1 + \gamma_{hk} g^2 + \delta_{hk} g^3, \qquad h = 1, 2, 3; \; k = 1, 2, 3. \tag{2}$$

Since the matrix $\left\| \dfrac{\partial g_i}{\partial u_j} \right\|$ is of rank three, there exists at least one determinant of third order in the matrix which is not zero. Let

$$\begin{vmatrix} g_1^1 & g_2^1 & g_3^1 \\ g_1^2 & g_2^2 & g_3^2 \\ g_1^3 & g_2^3 & g_3^3 \end{vmatrix} \neq 0.$$

If we now take $g_1$, $g_2$ and $g_3$ for independent variables, it follows from equations (2) that

$$\frac{\partial^2 g_i}{\partial g_h \partial g_k} = 0, \qquad i = 1, 2, \ldots, n; \; h = 1, 2, 3; \; k = 1, 2, 3.$$

Hence,

$$g_i = a_i g_1 + b_i g_2 + c_i g_3 + d_i, \qquad i = 1, 2, \ldots, n,$$

where $a_i$, $b_i$, etc., are constants. Hence, the three-spread lies in an $S_3$.

7. Through each point of a three-spread which satisfies *five* independent homogeneous linear partial differential equations of the second order pass an infinite number of tangents having contact of the second order with the three-spread. For, let

$$\Omega_i = \sum_{h,k=1}^{3} \omega_{hk} f_i^{hk}.$$

By a suitable choice of the quantities $\omega_{hk}$, it is now possible to replace the five given equations by the following six:

$$f^{hk} = a_{hk} f + b_{hk} f^1 + c_{hk} f^2 + d_{hk} f^3 + e_{hk} \Omega, \qquad h = 1, 2, 3; \; k = 1, 2, 3. \tag{3}$$

The three-point tangents at $(u_1, u_2, u_3)$ are those determined by values of the ratios $du_1 : du_2 : du_3$ which reduce to four the rank of the matrix:

$$\left\| \begin{matrix} f_0 & f_1 & \cdots & f_n \\ f_0^1 & f_1^1 & \cdots & f_n^1 \\ f_0^2 & f_1^2 & \cdots & f_n^2 \\ f_0^3 & f_1^3 & \cdots & f_n^3 \\ \phi_0 & \phi_1 & \cdots & \phi_n \end{matrix} \right\|, \tag{4}$$

in which

$$\phi_i = \sum_{h=1}^{3} \sum_{k=1}^{3} f_i^{hk} \, du_h \, du_k, \qquad i = 0, 1, \ldots, n.$$

Substituting for the functions $f_i^{hk}$ their values from equations (3) and subtracting multiples of the first four rows of the matrix, it is found that $\phi_i$ may be replaced by $\theta_i$, where

$$\theta_i = \Omega_i \sum_{h=1}^{3} \sum_{k=1}^{3} e_{hk}\, du_h\, du_k, \qquad i = 0, 1, \ldots, n.$$

Since, for all values of $i$, $\theta_i$ contains the factor

$$\sum_{h=1}^{3} \sum_{k=1}^{3} e_{hk}\, du_h\, du_k, \tag{5}$$

any set of values of $du_1 : du_2 : du_3$ which annuls this factor reduces the rank of the matrix to four. Hence, any tangent at $(u_1, u_2, u_3)$ determined by such a set of values of $du_1 : du_2 : du_3$ is a three-point tangent at $(u_1, u_2, u_3)$ to the three-spread.

8. Since the factor (5) is quadratic in $(du_1, du_2, du_3)$, the three-point tangents at each point $(u_1, u_2, u_3)$ of the three-spread form a quadric cone. It follows that the three-spread is either an hypersurface in $S_4$ or is formed by a system of planes such that consecutive planes intersect in a line.

For, the section of the three-spread by an arbitrary hyperplane is a two-spread having, at each point, two three-point tangents. SEGRE has shown* that such a two-spread either lies in an $S_3$ or is a developable or cone.

9. If the section of the three-spread by an arbitrary hyperplane is a two-spread lying in an $S_3$, then the three-spread lies in an $S_4$. For, if not, let $P_1$, $P_2, \ldots, P_6$ be six points of the three-spread which determine an $S_5$. An hyperplane passing through $P_1, P_2, \ldots, P_5$, and not passing through $P_6$, intersects the three-spread in a two-spread not lying in an $S_3$.

10. If the two-spread of section by an arbitrary hyperplane is a developable or cone, the two three-point tangents at each point of the two-spread are consecutive and lie entirely on the two-spread. Hence, the cone of three-point tangents to the three-spread at an arbitrary point degenerates into a double plane lying entirely on the three-spread. The three-spread is therefore generated by a simply infinite system of planes. Consecutive planes of this system intersect in a line, since the system of planes is intersected by an arbitrary hyperplane in a system of lines such that consecutive lines intersect. Such a three-spread is generated by either (a) the osculating planes to a curve, or

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* *Loc. cit.*, p. 571.

(b) the planes projecting the tangents to a curve from a fixed point, or (c) the planes projecting the points of a curve from a fixed line.

11. Conversely, let

$$x_i = f_i(u_1, u_2, u_3), \qquad i = 0, 1, 2, 3, 4,$$

be the equations of an hypersurface in an $S_4$. The condition that these five functions $f_i$ satisfy an homogeneous linear partial differential equation of the second order is only five conditions on the ten coefficients of the differential equation. There exist, therefore, five such equations satisfied by each of the five functions. Any such three-spread, therefore, satisfies five such equations.

Again, consider a three-spread the equations of which can be put into one of the forms

$$\begin{aligned} x_i &= f_i(u_1, u_2, u_3) = g_i(u_3) + u_1 g_i^3(u_3) + u_2 g_i^{33}(u_3), && i = 0, 1, \ldots, n; \\ x_i &= f_i(u_1, u_2, u_3) = g_i(u_3) + u_1 g_i^3(u_3) + u_2 k_i, && i = 0, 1, \ldots, n; \\ x_i &= f_i(u_1, u_2, u_3) = g_i(u_3) + u_1 l_i + u_2 k_i, && i = 0, 1, \ldots, n, \end{aligned}$$

in which $k_i$ and $l_i$ are constants.

In each case, the three-spread satisfies the equations:

$$\begin{aligned} &f^{11} = 0, \qquad f^{12} = 0, \qquad f^{13} = 0, \\ &f^{13} = af + bf^1 + cf^2 + df^3, \\ &f^{23} = a_1 f + b_1 f^1 + c_1 f^2 + d_1 f^3. \end{aligned}$$

Hence, *the necessary and sufficient condition that a three-spread satisfy five homogeneous linear partial differential equations of the second order is that it be either (a) an hypersurface in an $S_4$ or (b) generated by planes in such a way that consecutive planes intersect in a line.*

## On Three-Spreads Which Satisfy Four Homogeneous Linear Partial Differential Equations of the Second Order.

12. Through an arbitrary point of a three-spread which satisfies four homogeneous linear partial differential equations of the second order, pass four lines having contact of the second order with the three-spread. For, let

$$\Omega_i = \sum_{h=1}^{3} \sum_{k=1}^{3} \omega_{hk} f_i^{hk}, \qquad \Psi_i = \sum_{h=1}^{3} \sum_{k=1}^{3} \psi_{hk} f_i^{hk}, \qquad i = 0, 1, \ldots, n.$$

By a suitable choice of $\omega_{hk}$ and $\psi_{hk}$ the four given equations may be replaced by

$$f^{hk} = a_{hk} + b_{hk} f^1 + c_{hk} f^2 + d_{hk} f^3 + e_{hk} \Psi + g_{hk} \Omega, \qquad h, k = 1, 2, 3. \quad (1)$$

Hence, in the matrix which determines the three-point tangents (¶ 7), $\phi_i$ may be replaced by

$$\theta_i = \Psi_i \sum_{h=1}^{3} \sum_{k=1}^{3} e_{hk}\, du_h\, du_k + \Omega_i \sum_{h=1}^{3} \sum_{k=1}^{3} g_{hk}\, du_h\, du_k .$$

Hence, the lines through the point whose parameters are $(u_1, u_2, u_3)$, for which $du_1 : du_2 : du_3$ satisfy

$$E = \sum_{h=1}^{3} \sum_{k=1}^{3} e_{hk}\, du_h\, du_k = 0 \quad \text{and} \quad G = \sum_{h=1}^{3} \sum_{k=1}^{3} g_{hk}\, du_h\, du_k = 0, \tag{2}$$

are three-point tangents to the three-spread. Solving equations (2) for the ratios $du_1 : du_2 : du_3$ and integrating, we find four systems of $\infty^2$ curves on the three-spread the tangents to which are three-point tangents to the three-spread. These curves correspond, on the three-spreads of the types under consideration, to the asymptotic lines on a surface in $S_3$.

13. By a transformation of the curvilinear coordinates $(u_1, u_2, u_3)$, we may take one of the four systems of curves whose tangents are three-point tangents to the three-spread to be the curves $u_2 =$ const., $u_3 =$ const. Suppose this transformation effected. It then follows that

$$f^{11} = a_{11} f + b_{11} f^1 + c_{11} f^2 + d_{11} f^3. \tag{3}$$

Suppose, first, that the three-spread does *not* satisfy any equation of the form

$$af + bf^1 + cf^2 + df^3 + ef^{12} + gf^{13} = 0. \tag{4}$$

It then follows that the remaining three equations satisfied by the three-spread can be written in the form:

$$f^{22} = a_{22} f + b_{22} f^1 + c_{22} f^2 + d_{22} f^3 + e_{22} f^{12} + g_{22} f^{13}, \tag{5}$$
$$f^{23} = a_{23} f + b_{23} f^1 + c_{23} f^2 + d_{23} f^3 + e_{23} f^{12} + g_{23} f^{13}, \tag{6}$$
$$f^{33} = a_{33} f + b_{33} f^1 + c_{33} f^2 + d_{33} f^3 + e_{33} f^{12} + g_{33} f^{13}. \tag{7}$$

Differentiating equations (3), (5), (6) and (7) and substituting the values of $f^{11}, f^{22}, f^{23}$ and $f^{33}$ from these equations, it is seen that all the derivatives of order higher than the second can be expressed in terms of $f, f^1, f^2, f^3, f^{12}$ and $f^{13}$.

Let $(u_1, u_2, u_3)$ be given a fixed set of values which determine an arbitrary point on the three-spread. Since the three-spread does not satisfy an equation of the form (4), the six points

$$x_i = f_i, \; x_i = f_i^1, \; x_i = f_i^2, \; x_i = f_i^3, \; x_i = f_i^{12}, \; x_i = f_i^{13}, \qquad i = 0, 1, \ldots, n, \tag{8}$$

determine an $S_5$. The three-spread lies in this $S_5$. For, let $u_1 = \phi_1(t)$, $u_2 = \phi_2(t)$, $u_3 = \phi_3(t)$ determine an analytic curve $C$ on the three-spread.

Since the successive derivatives of $f_i(t)$ with respect to $t$ may be expressed linearly and homogeneously in terms of $f_i, f_i^1, f_i^2, f_i^3, f_i^{12}, f_i^{13}$, $C$ has, at each point $P$ on it, contact of higher than the fifth order with the $S_5$ (8) corresponding to $P$. Hence, these $S_5$ coincide and contain $C$. It follows that the three-spread lies in an $S_5$. For, if not, seven points taken at random on it would not lie in an $S_5$. But through any seven such points an analytic curve $C$ can be passed. They therefore lie in an $S_5$.

If the three-spread does satisfy an equation of the type (4), it follows from equation (2) that $du_2 = du_3 = 0$ counts twice, at least, as a solution of $E = 0$ and $G = 0$. The tangents to $u_2 =$ const., $u_3 =$ const. therefore count twice as three-point tangents to the three-spread. Hence, *if through an arbitrary point of the three-spread passes a tangent to the three-spread whose direction is determined by a set of values of $du_1 : du_2 : du_3$ which count only once as solutions of equations* (2), *then the three-spread lies in an $S_5$.*

It will be shown hereafter that if, at an arbitrary point, every solution of (2) counts twice, at least, as a solution, then the three-spread may lie in a space of any number of dimensions.

14. Through each point of the three-spread pass, in general, in addition to the four curves determined by equations (2), three other curves of especial importance. To derive them, notice, first, that the section of the three-spread by an hyperplane through the tangent $S_3$ at an arbitrary point $P$ is a two-spread having a node at $P$. The tangent quadric cone at $P$ to this two-spread lies in the tangent $S_3$ to the three-spread and contains the four three-point tangents to the three-spread at $P$. If the hyperplane varies in such a way as always to contain the tangent $S_3$, this tangent cone therefore describes a pencil of cones through the three-point tangents. The cones of this pencil are generated by the lines through $P$ determined by

$$\sum_{h=1}^{3} \sum_{k=1}^{3} e_{hk}\, du_h\, du_k + \lambda \sum_{h=1}^{3} \sum_{k=1}^{3} g_{hk}\, du_h\, du_k = 0,$$

where $\lambda$ is a parameter the variation of which determines the different cones of the pencil.

Three cones of this pencil are composite. Their double lines pass through $P$. Hence, *through $P$ pass three tangents to the three-spread which are double lines of tangent quadric cones at $P$ to sections of the three-spread by hyperplanes through the tangent $S_3$ at $P$.* These three tangents will be referred to as the three "funda-

mental tangents" at $P$. To determine the direction of one of these tangents, it is necessary to determine a root $\lambda$ of

$$| e_{hk} + \lambda g_{hk} | = 0,$$

and then determine the ratios $du_1 : du_2 : du_3$ from

$$\sum_{h=1}^{3} (e_{hk} + \lambda g_{hk})\, du_h = 0, \qquad k = 1, 2, 3. \tag{9}$$

Integrating these three equations for each value of $\lambda$, we obtain three systems $\infty^2$ of curves on the three-spread whose tangents are fundamental tangents to the three-spread.

15. Let $(u_1 + du_1,\ u_2 + du_2,\ u_3 + du_3)$ be a point of the three-spread consecutive to $P$ in any direction. *The tangent $S_3$ at this consecutive point intersects the tangent $S_3$ at $P$ in a line through $P$ the direction of which corresponds to the direction determined by $du_1 : du_2 : du_3$ in an involutorial birational quadratic transformation of which the coincidences are the three-point tangents at $P$ and the fundamental lines are the three fundamental tangents at $P$.*

For, the tangent $S_3$ at the consecutive point is determined, to infinitesimals of the second order, by the four points:

$$\left.\begin{aligned} x_i &= f_i + \sum_{h=1}^{3} f_i^h\, du_h, && i = 0, 1, \ldots, n; \\ x_i &= f_i^k + \sum_{h=1}^{3} f_i^{hk}\, du_h, && k = 1, 2, 3;\ i = 0, 1, \ldots, n. \end{aligned}\right\} \tag{10}$$

From equations (1) we deduce:

$$\left.\begin{aligned} f_i^k + \sum_{h=1}^{3} f_i^{hk}\, du_h = f_i^k + f_i \sum_{h=1}^{3} a_{hk}\, du_h + f_i^1 \sum_{h=1}^{3} b_{hk}\, du_h + f_i^2 \sum_{h=1}^{3} c_{hk}\, du_h \\ + f_i^3 \sum_{h=1}^{3} d_{hk}\, du_h + \Psi_i \sum_{h=1}^{3} e_{hk}\, du_h + \Omega_i \sum_{h=1}^{3} g_{hk}\, du_h, \\ k = 1, 2, 3;\ i = 0, 1, \ldots, n. \end{aligned}\right\} \tag{11}$$

From (10) the equations of the tangent $S_3$ at the consecutive point may be written, in terms of the parameters $(v_1, v_2, v_3)$, in the form:

$$x_i = f_i + \sum_{h=1}^{3} f_i^h\, du_h + \sum_{k=1}^{3} v_k \left(f_i^k + \sum_{h=1}^{3} f_i^{hk}\, du_h\right), \qquad i = 0, 1, \ldots, n.$$

Substituting into this expression the values of $f_i^k + \sum_{h=1}^{3} f_i^{hk}\, du_h$ from (11), it is

seen that the points which lie in the tangent $S_3$ at the consecutive point and also in the tangent $S_3$ at $P$, *i. e.* in the $S_3$ determined by the four points

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \qquad i = 0, 1, \ldots, n,$$

are those which satisfy the equations:

$$\left.\begin{aligned} v_1 \sum_{h=1}^{3} e_{1h}\, du_h + v_2 \sum_{h=1}^{3} e_{2h}\, du_h + v_3 \sum_{h=1}^{3} e_{3h}\, du_h = 0, \\ v_1 \sum_{h=1}^{3} g_{1h}\, du_h + v_2 \sum_{h=1}^{3} g_{2h}\, du_h + v_3 \sum_{h=1}^{3} g_{3h}\, du_h = 0. \end{aligned}\right\} \tag{12}$$

Since these two equations are linear, the two $S_3$ intersect in a straight line. Since the equations are satisfied when $v_1 = v_2 = v_3 = 0$, the line goes through the point $(u_1 + du_1,\ u_2 + du_2,\ u_3 + du_3)$. In the limit, therefore, as the consecutive point approaches $P$ in the direction $du_1 : du_2 : du_3$, the line of intersection of the two consecutive tangent $S_3$ approaches the line through $P$ in the direction determined by the values of the ratios $\delta u_1 : \delta u_2 : \delta u_3$ which satisfy

$$\sum_{h=1}^{3} \sum_{k=1}^{3} e_{hk}\, du_h\, \delta u_k = 0, \qquad \sum_{h=1}^{3} \sum_{k=1}^{3} g_{hk}\, du_h\, \delta u_k = 0. \tag{13}$$

But the line so determined is the intersection of the polars of the line $du_1 : du_2 : du_3$ with respect to the quadric cones determined by equations (2). The correspondence between the two directions at $P$ is therefore an involutorial birational correspondence. The four united lines of the correspondence are the four intersections of the cones $E = 0$ and $G = 0$, *i. e.* the three-point tangents at $P$. The three fundamental lines of the correspondence are the three double lines of the pencil $E + \lambda\, G = 0$, *i. e.* the three fundamental tangents at $P$.

15. If, in particular, a consecutive point is taken in the direction of any one of the fundamental tangents, then the tangent $S_3$ at the consecutive point intersects the tangent $S_3$ at $P$ in the plane joining the other two fundamental tangents. It therefore lies in an $S_4$ through the tangent $S_3$ at $P$. It is, in fact, easily seen that, if $du_1 : du_2 : du_3$ is a solution of equations (9), then the tangent $S_3$ at $(u_1 + du_1,\ u_2 + du_2,\ u_3 + du_3)$ lies, to infinitesimals of the second order, in the $S_4$ determined by

$$x_i = f_i, \; x_i = f_i^1, \; x_i = f_i^2, \; x_i = f_i^3, \; x_i = \Omega_i - \lambda\, \Psi_i, \qquad i = 0, 1, \ldots, n.$$

The tangent $S_3$ to the three-spread along an integral curve of (9), therefore, form, in general, the osculating $S_3$ to a curve $C$. Each osculating plane to $C$ touches the three-spread at a point $P$ of the integral curve of equations (9) and

contains the other two fundamental tangents to the three-spread at the point of tangency. The tangent lines to $C$ do not, in general, touch the three-spread. In fact, since the tangent lines to $C$ are the lines of intersection of the tangent $S_3$ to the three-spread at three consecutive points of the integral curve of equations (9), if the tangent to $C$ passed through $P$, the tangent $S_3$ at three consecutive points of the integral curve of (9) would pass through $P$. The tangent at $P$ to the integral curve of (9) would therefore be a three-point tangent at $P$ to the three-spread. This is not, in general, the case.

We shall now discuss the behavior of the three-spread when the tangent cones to the sections of the three-spread by the hyperplanes through the tangent $S_3$ at an arbitrary point $P$ on the three-spread satisfy certain particular conditions. Consider, first, the case in which:

### *Case* 1. *The Tangent Cones at P Are All Composite.*

A. Let the three-spread be generated by planes.

16. If a three-spread is generated by planes, an hyperplane through the tangent $S_3$ at an arbitrary point $P$ on the three-spread intersects the three-spread in a two-spread the tangent cone to which, at $P$, has the generating plane through $P$ for a component. It is, therefore, certainly composite.

A three-spread generated by planes satisfies, in general, however, only three homogeneous linear partial differential equations of the second order. In fact, if the equations of the three-spread are reduced to the form

$$x_i = f_i(u_1, u_2, u_3) = f_{i1}(u_3) + u_1 f_{i2}(u_3) + u_2 f_{i3}(u_3), \qquad i = 0, 1, \ldots, n,$$

we have identically only

$$f^{11} = 0, \qquad f^{12} = 0, \qquad f^{22} = 0. \tag{1}$$

Impose, now, the condition that the three-spread satisfy a fourth homogeneous linear partial differential equation of the second order:

$$a_{33} f^{33} + a_{23} f^{23} + a_{13} f^{13} + a_3 f^3 + a_2 f^2 + a_1 f^1 + a f = 0. \tag{2}$$

Suppose, first, that $a_{33} \neq 0$. Then the tangent to the three-spread at an arbitrary point for which $du_1 : du_2 : du_3 = a_{13} : a_{23} : 2\,a_{33}$ counts only once as a three-point tangent. The three-spread therefore lies in an $S_5$.

Suppose, now, that $a_{33} = 0$. It then follows that consecutive planes of the three-spread lie in an $S_4$ and therefore intersect in a point. They therefore

either all pass through a fixed point or touch a fixed curve. In fact, equations (2) may, in this case, be written in the form

$$af_{i1} + bf_{i2} + cf_{i3} + df_{i1}^3 + ef_{i2}^3 + gf_{i3}^3 = 0, \qquad i = 0, 1, \ldots, n,$$

where $a$, $b$, etc., are independent of $u_1$ and $u_2$, and where $d$, $e$ and $g$ are not all zero, since $a_{23}$ and $a_{13}$ are not both zero. By a linear transformation of $u_1$ and $u_2$ this equation may be reduced to

$$f_{i3}^3 = \alpha_1 f_{i1} + \alpha_2 f_{i2} + \alpha_3 f_{i3}, \qquad i = 0, 1, \ldots, n.$$

If $\alpha_1 = \alpha_2 = 0$, this may still further be reduced to

$$f_{i3}^3 = 0, \qquad i = 0, 1, \ldots, n.$$

The planes, therefore, all pass through a point.

If $\alpha_1$ and $\alpha_2$ are not both zero, let $\alpha_2 \neq 0$. By a linear transformation of $u_1$ and $u_2$ the equation may be reduced to

$$f_{i3}^3 = f_{i2}, \qquad i = 0, 1, \ldots, n.$$

The equations of the three-spread now reduce to

$$x_i = f_{i1}(u_3) + u_1 f_{i3}^3(u_3) + u_2 f_{i3}(u_3), \qquad i = 0, 1, \ldots, n.$$

The generating planes, therefore, all touch the curve $x_i = f_{i3}(u_3)$. The three-spread does not, in general, lie in an $S_5$.

17. Conversely, if a three-spread generated by planes lies in an $S_5$, it satisfies four homogeneous linear partial differential equations of the second order, since, as we have seen, any three-spread lying in an $S_5$ satisfies four such equations. If a three-spread is generated by planes passing through a fixed point, its equations may be put into the form

$$x_i = f_i(u_3) + u_1 f_{i2}(u_3) + u_2 k_i, \qquad i = 0, 1, \ldots, n,$$

where the quantities $k_i$ are constants. It then satisfies the four equations:

$$f^{11} = 0, \quad f^{12} = 0, \quad f^{22} = 0, \quad f^{23} = 0.$$

Finally, if the three-spread is generated by planes touching a fixed curve, its equations may be put into the form

$$x_i = f_{i1}(u_3) + u_1 f_{i3}^3(u_3) + u_2 f_{i3}(u_3), \qquad i = 0, 1, \ldots, n.$$

It then satisfies the equations:

$$f^{11} = f^{12} = f^{22} = f^{23} - f^1 = 0.$$

Hence, *the necessary and sufficient condition that a three-spread generated by planes satisfy four homogeneous linear partial differential equations of the second*

*order is that it either (a) lie in an $S_5$, or (b) be generated by planes all passing through a fixed point, or (c) be generated by planes all touching a fixed curve.*

18. For the three-spreads of case (a) for which the coefficient $a_{33}$ of equation (2) does not vanish, the birational correspondence considered above between the tangents at an arbitrary point $P$ reduces to an involutorial projectivity having the generating plane through $P$ for united plane and the discrete three-point tangent for united tangent. It follows, in fact, from equations (1) and (2) that the tangent $S_3$ at a point consecutive to $P$ in the direction $du_1 : du_2 : du_3$ intersects the tangent $S_3$ at $P$ in a line through $P$ in a direction determined by $\delta u_1 : \delta u_2 : \delta u_3$, where

$$\delta u_1 = a_{13}\, du_3 - a_{33}\, du_1, \quad \delta u_2 = a_{23}\, du_3 - a_{33}\, du_2, \quad \delta u_3 = a_{33}\, du_3.$$

For the three-spreads of cases (b) and (c) it is similarly seen, since $a_{33} = 0$, that the correspondence is degenerate. To every direction through $P$ corresponds the line joining $P$ to the point of intersection of the plane through $P$ with the consecutive plane.

B. The three-spread is not generated by planes.

19. The tangent cone at an arbitrary point $P$ to the section of the three-spread by an arbitrary hyperplane through the tangent $S_3$ at $P$ has a double line, since, by hypothesis, it is composite. Moreover, since the three-spread is not generated by planes, through $P$ there does not pass a continuum of straight lines lying on the three-spread. For, suppose that through $P$ there could pass such a continuum of lines. Since the three-spread is not generated by planes, the continuum could not lie in a plane. Moreover, the continuum, being supposed to lie on the three-spread, would lie upon the tangent cone at $P$ to the section of the three-spread by an arbitrary hyperplane through the tangent $S_3$ at $P$. This tangent cone would therefore be invariant for every such hyperplane. This is impossible, since the three-spread satisfies only four homogeneous linear partial differential equations of the second order.

The three-spread is therefore generated by straight lines such that the tangent $S_3$ along each straight line is invariant.

Conversely, if a three-spread is generated by straight lines in such a way that the tangent $S_3$ is invariant along each straight line, then the section of the three-spread by an hyperplane through the tangent $S_3$ at an arbitrary point $P$ has as a double line the line through $P$ along which the tangent $S_3$ is invariant. The tangent cone at $P$ to the section is therefore composed of two planes through this double line. It is therefore composite.

Hence, the necessary and sufficient condition that the tangent cone at $P$ to the two-spread of section of a three-spread not generated by planes, by an arbitrary hyperplane through the tangent $S_3$ at $P$, be composite, is that the three-spread be generated by straight lines in such a way that the tangent $S_3$ is invariant along each line.

20. Let

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 h_i(u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

be such a three-spread, the tangent $S_3$ being invariant along the lines $u_2 = \text{const.}$, $u_3 = \text{const.}$ We then have:

$$f^{11} = 0, \tag{1}$$
$$f^{12} = af + bf^1 + cf^2 + df^3, \tag{2}$$
$$f^{13} = a_1 f + b_1 f^1 + c_1 f^2 + d_1 f^3. \tag{3}$$

Differentiating (2) with respect to $u_3$ and (3) with respect to $u_2$ and subtracting, we obtain:

$$c_1 f^{22} + (d_1 - c) f^{23} - d f^{33} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3. \tag{4}$$

Such a three-spread therefore satisfies four homogeneous linear partial differential equations of the second order, unless $c_1 = d_1 - c = d = 0$.

21. Suppose, however, that $c_1 = d_1 - c = d = 0$. It follows that all the coefficients of (4) vanish, since the three-spread can not satisfy an equation of the first order. If we differentiate (2) and (3) with respect to $u_1$, the resulting equations must be proportional to $u_1$. Hence, (2) and (3) may be written in the form

$$\alpha h^2 + \beta g^2 + \gamma h + \delta g = 0,$$
$$\alpha h^3 + \beta g^3 + \gamma_1 h + \delta_1 g = 0,$$

where $\alpha$, $\beta$, $\gamma$, etc., are independent of $u_1$. Moreover, $\alpha$ and $\beta$ are not both zero, since, otherwise, equations (2) and (3) would reduce to equations of the first order. By a transformation of curvilinear coordinates which is linear in $u_1$, these two equations may therefore be reduced to the form:

$$h^2 = \xi g + \eta h, \qquad h^3 = \xi_1 g + \eta_1 h. \tag{5}$$

Differentiating the first of these equations with respect to $u_3$, the second with respect to $u_2$, and subtracting, we obtain:

$$\xi g^3 - \xi_1 g^2 = \sigma g + \tau h.$$

Hence $\xi = \xi_1 = 0$. For, otherwise, the three-spread would satisfy an homogeneous linear partial differential equation of the first order. Equations (5),

therefore, reduce to $h^2 = \eta h$ and $h^3 = \eta_1 h$. Hence the $n+1$ functions $h_i$ are of the form $h_i = k_i \phi(u_2, u_3)$, where the quantities $k_i$ are constants. Hence, by a slight transformation of curvilinear coordinates which is linear in $u_1$, the functions $h_i$ may be reduced to constants. The three-spread is therefore a cone.

Let the quantities $h_i$ be reduced to constants, and suppose that, in addition to equations (1), (2) and (3), there exists a fourth homogeneous linear partial differential equation of the second order which is satisfied by the three-spread. This equation reduces at once to

$$A g^{22} + B g^{23} + C g^{33} = a_2 g + b_2 g^2 + c_2 g^3 + d_2 h. \tag{4'}$$

The coefficients of this equation must be independent of $u_1$, since, otherwise, the three-spread would satisfy more than four such equations.

Let $\theta$ be a solution of the equation

$$A \theta^{22} + B \theta^{23} + C \theta^{33} = a_2 \theta + b_2 \theta^2 + c_2 \theta^3 + d_2.$$

If we replace $u_1$ by $u_1 - \theta$ in the equation of the three-spread, equation (4′) reduces at once to

$$A g^{22} + B g^{23} + C g^{33} = a_2 g + b_2 g^2 + c_2 g^3. \tag{4''}$$

Hence the three-spread is the projection from a fixed point of a two-spread which satisfies an homogeneous partial differential equation of the second order.

Conversely, consider a three-spread cone,

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 h_i, \qquad i = 0, 1, \ldots, n,$$

which projects from a fixed point $x_i = h_i$ a two-spread which satisfies an equation of the type (4″). It is seen at once that the three-spread satisfies the four equations:

$$\begin{gathered} f^{11} = 0, \qquad f^{12} = 0, \qquad f^{13} = 0, \\ A f^{22} + B f^{23} + C f^{33} = a_2 f + b_2 f^2 + c_2 f^3 - a_2 u_1 f^1. \end{gathered} \tag{4''}$$

We have therefore proved that: *A three-spread generated by straight lines in such a way that the tangent $S_3$ along each line of the system is invariant satisfies four homogeneous linear partial differential equations of the second order, unless it is a cone. If the three-spread is a cone, it satisfies four such equations if, and only if, it is the projection from a fixed point of a two-spread satisfying an homogeneous linear partial differential equation of the second order.* In neither case does it lie, in general, in an $S_5$.

21. On a three-spread which satisfies equations (1), (2), (3) and either (4) or (4‴), the birational correspondence between the tangents at an arbitrary

point $P$ is composite. To any direction through $P$ corresponds the direction of the invariant line through $P$. For, if we write (4) or (4''') in the form

$$\alpha_{22} f^{22} + \alpha_{23} f^{23} + \alpha_{33} f^{33} = \alpha f + \alpha_1 f^1 + \alpha_2 f^2 + \alpha_3 f^3,$$

and if we choose $\beta_{22}$, $\beta_{23}$, $\beta_{33}$, $\gamma_{22}$, $\gamma_{23}$ and $\gamma_{33}$ so as to make the determinant

$$| \, \alpha_{22} \; \beta_{23} \; \gamma_{33} \, |$$

different from zero, we find, for the direction through $P$ corresponding to $du_1 : du_2 : du_3$, the intersection of the polars of $(du_1,\ du_2,\ du_3)$ with respect to

$$(\alpha_{33}\beta_{23} - \alpha_{23}\beta_{33})\, du_2^2 + 2\,(\alpha_{22}\beta_{33} - \alpha_{33}\beta_{22})\, du_2\, du_3 + (\alpha_{23}\beta_{22} - \alpha_{22}\beta_{23})\, du_3^2 = 0,$$
$$(\alpha_{33}\gamma_{23} - \alpha_{23}\gamma_{33})\, du_2^2 + 2\,(\alpha_{22}\gamma_{33} - \alpha_{33}\gamma_{22})\, du_2\, du_3 + (\alpha_{23}\beta_{33} - \alpha_{22}\beta_{23})\, du_3^2 = 0.$$

The direction required is therefore determined by

$$\left.\begin{aligned} &[(\alpha_{33}\beta_{23} - \alpha_{23}\beta_{33})\, du_2 + (\alpha_{22}\beta_{33} - \alpha_{33}\beta_{22})\, du_3]\, \delta u_2 + [(\alpha_{22}\beta_{33} - \alpha_{33}\beta_{22})\, du_2 + (\alpha_{23}\beta_{22} - \alpha_{22}\beta_{23})\, du_3]\, \delta u_3 = 0, \\ &[(\alpha_{33}\gamma_{23} - \alpha_{23}\gamma_{33})\, du_2 + (\alpha_{22}\gamma_{33} - \alpha_{33}\gamma_{22})\, du_3]\, \delta u_2 + [(\alpha_{22}\gamma_{33} - \alpha_{33}\gamma_{22})\, du_2 + (\alpha_{23}\gamma_{22} - \alpha_{22}\gamma_{23})\, du_3]\, \delta u_3 = 0. \end{aligned}\right\} \quad (6)$$

Hence, in general, $\delta u_2 = \delta u_3 = 0$.

If, however, $du_2$ and $du_3$ satisfy the condition

$$\alpha_{22}\, du_3^2 - \alpha_{23}\, du_2\, du_3 + \alpha_{33}\, du_2^2 = 0, \qquad (7)$$

equations (6) become identical. The tangent $S_3$ at the consecutive point $(u_1 + du_1,\ u_2 + du_2,\ u_3 + du_3)$ now intersects the tangent $S_3$ at $P$ in a plane through the invariant line $du_2 = du_3 = 0$ and through the tangent to the two-spread $u_1 = \text{const.}$ determined by (6).

Moreover, if $du_2 : du_3$ is one solution of (7), then the value of $\delta u_2 : \delta u_3$ corresponding to it in equation (6) is the other solution of (7). Hence, *if $P$ is an arbitrary point of the three-spread, two distinct or coincident planes pass through the invariant line through $P$ such that, if $P'$ is consecutive to $P$ in one of these planes, the tangent $S_3$ at $P'$ intersects the tangent $S_3$ at $P$ in the other plane.*

22. *These planes are invariant as $P$ moves along the invariant line.* For, differentiating equation (4) or (4''') with respect to $u_1$, we find, since the three-spread satisfies only four homogeneous linear partial differential equations of the second order, that the ratios $\alpha_{22} : \alpha_{23} : \alpha_{33}$ are independent of $u_1$. Hence, equation (7) is independent of $u_1$.

If the three-spread is a cone, the above theorem follows at once from known properties of two-spreads satisfying an homogeneous linear partial differential equation of the second order.*

* Segre, *loc. cit.*, p. 576.

23. We shall now distinguish two cases according to the equality or inequality of the roots of equation (7). Suppose first that the roots are unequal.

If the three-spread is a cone, it is, in this case, simply the projection of a two-spread having two distinct systems of "characteristic lines."*

If the three-spread is not a cone, it is generated in two ways by a system $\infty^1$ of cones or of developables whose generators are the invariant lines of the three-spread and whose edges of regression form a system of characteristic lines on a two-spread lying on the given three-spread.

For, since equation (7) is independent of $u_1$, and since the roots of (7) are unequal, they may, by a transformation of the parameters $u_2$ and $u_3$, be reduced to $du_2 = 0$ and $du_3 = 0$. We then have, in equation (4), $c_1 = d = 0$. Hence, from equations (1) and (2) it follows that the two-spreads $u_3 =$ const. are developables or cones. Similarly, from (1) and (3) it follows that the two-spreads $u_2 =$ const. are developables or cones.

If both systems are cones, the equations of the three-spread may be reduced to

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2) + u_1 h_i(u_3), \qquad i = 0, 1, \ldots, n.$$

The three-spread is therefore generated by the lines cutting each of two curves.

Conversely, any three-spread generated by the common secants to two curves, has its tangent $S_3$ invariant along the secant lines. For, putting the equations of the three-spread in the above form, we obtain:

$$f^{11} = f^{12} = f^{23} = f^3 - u_1 f^{13} = 0.$$

If the two-spreads of at least one of the two systems are developables, the equations of the three-spread may be put into the form:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 g_i^3(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

From equation (2) it follows that

$$g^{23} = \alpha g + \beta g^2 + \gamma g^3. \tag{8}$$

Hence, the two-spread,

$$x_i = g_i(u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

formed by the edges of regression of the developables $u_2 =$ const., has two distinct systems of characteristic curves. From the form of equation (8) it is seen that one of these systems of characteristic curves is the system $u_2 =$ const.,

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† The "*caratteristiche*" of Segre.

*i. e.* the edges of regression themselves. The other system is formed by the curves $u_3 = \text{const.}$ These are the curves in which the generators of the second system of developables or cones touch the two-spread. Hence, *the generators of the three-spread which touch the two-spread along a characteristic curve of the second system generate a developable or cone.*

Conversely, let

$$x_i = g_i(u_2, u_3), \qquad i = 0, 1, \dots, n,$$

be a two-spread having two distinct systems of characteristic curves, and let $u_2 = \text{const.}$ and $u_3 = \text{const.}$ determine these two systems of curves so that

$$g^{23} = \alpha g + \beta g^2 + \gamma g^3.$$

Then the three-spread generated by the tangents to either system of characteristic curves, for example, the three-spread

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 g_i^3(u_2, u_3), \qquad i = 0, 1, \dots, n,$$

satisfies the four equations

$$\begin{aligned} f^{11} &= 0, \\ f^{12} &= af + bf^1 + cf^2 + df^3, \\ f^{13} &= a_1 f + b_1 f^1 + c_1 f^2 + d_1 f^3, \\ f^{23} &= a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3, \end{aligned}$$

and therefore has the lines $u_2 = \text{const.}$, $u_3 = \text{const.}$ for invariant lines. From the form of the last equation, the roots of equation (7) are seen to be distinct.

Hence, *the necessary and sufficient condition that a three-spread be generated by lines in such a way that the tangent $S_3$ along each line is invariant and that the two planes of intersecton of this $S_3$ with the tangent $S_3$ along consecutive generators are distinct is that the three-spread* (*a*) *be the projection from a fixed point of a two-spread having two distinct systems of characteristic curves, or* (*b*) *be generated by the lines intersecting each of two fixed curves, or* (*c*) *be generated by the tangents to a system of characteristic curves on a two-spread having two distinct systems of such curves.*

24. Suppose, next, that the roots of (7) are equal. If this condition is satisfied, the cones which intersect in the three-point tangents at $P$ have a common component. The plane so determined is a component of the tangent cone at $P$ to the section of the three-spread by every hyperplane through the tangent $S_3$ at $P$. Every line in the plane through $P$ is, therefore, a three-point tangent at $P$. Since this plane is invariant as $P$ moves along the invariant line, every line in the plane has three-point contact with the three-spread at its intersection with the invariant line.

Conversely, if a three-spread which does not satisfy more than four homogeneous linear partial differential equations of the second order has, at each point, a continuum of three-point tangents, these must form a pencil, since, otherwise, the section by an arbitrary hyperplane would lie in an $S_3$. The plane of this pencil must be a component of the tangent cone at its vertex to the section of the three-spread by any hyperplane through the tangent $S_3$ at the vertex. Hence, the three-spread, since it is not generated by planes, is generated by lines in such a way that the tangent $S_3$ is invariant along each line, and also in such a way that the roots of equation (7) are equal.

If the three-spread is a cone, the condition that the roots of (7) be equal reduces, when the values of $\alpha_{22}$, $\alpha_{23}$ and $\alpha_{33}$ are substituted in from equation (4'''), to

$$B^2 - 4AC = 0.$$

This, however, is the condition that the two-spread $x_i = g_i(u_2, u_3)$ have, at each point, a three-point tangent. A three-spread cone, therefore, will have, at each point, a pencil of three-point tangents if, and only if, it is the projection of a two-spread having, at each point, a three-point tangent.

If the three-spread is not a cone, the condition that the roots of (7), *i. e.* of (4), be equal reduces to

$$(d_1 - c)^2 + 4c_1 d = 0.$$

Since the ratios $c_1 : d_1 - c : -d$ are independent of $u_1$, equation (4) may be reduced to the form:

$$f^{22} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3.$$

We now have $d = 0$, $d_1 = c$. From equations (1) and (2), therefore, it follows that the two-spreads $u_3 =$ const. are developables or cones. They can not be cones; for, if they were, the three-spread itself would be conical, or else generated by planes. They are therefore developables. The equations of the three-spread may, therefore, be put into the form

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 g_i^2(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

From equations (2) and (3) we now obtain

$$g^{22} = \alpha g + \beta g^2 + \gamma g^3.$$

Hence the edges of regression of the developables $u_3 =$ const. are three-point tangent curves on the two-spread:

$$x_i = g_i(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

Conversely, consider a three-spread generated by a system of three-point tangents to a two-spread. Let the equations of the three-spread be

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 g_i^2(u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

in which, since the generators of the three-spread are three-point tangents to the two-spread,

$$g^{22} = \alpha g + \beta g^2 + \gamma g^3.$$

It therefore follows that the three-spread satisfies the four equations:

$$\begin{aligned} f^{11} &= 0, \\ f^{12} &= \frac{1}{u_1} f^2 - \frac{1}{u_1} f^1, \\ f^{13} &= a_1 f + b_1 f^1 + c_1 f^2 + \frac{1}{u_1} f^3, \\ f^{22} &= a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3. \end{aligned}$$

Hence, *the necessary and sufficient condition that a three-spread not generated by planes nor lying in an $S_4$ have, at an arbitrary point, a pencil of three-point tangents is that it (a) be a cone projecting a two-spread which has, at each point, a three-point tangent, or (b) be generated by a system $\infty^2$ of three-point tangents to a two-spread.*

It will be supposed throughout, hereafter, that the tangent cones at $P$ to the sections of the three-spread by the hyperplanes through the tangent $S_3$ at $P$ are not all composite. Of the four generators of intersection of the cones of this pencil at $P$ two or more may be consecutive. It will next be determined under what conditions it may happen that:

*Case 2. Two Three-Point Tangents Are Consecutive.*

25. Let $u_2 = \text{const.}$, $u_3 = \text{const.}$ determine the system of curves the tangents to which count twice as three-point tangents to the three-spread. Since the cones $\sum_{h,k=1}^{3} e_{hk}\, du_h\, du_k = 0$ and $\sum_{h,k=1}^{3} g_{hk}\, du_h\, du_k = 0$ touch along $du_2 = 0$, $du_3 = 0$, two of the four differential equations satisfied by the three-spread may be written in the form:

$$f^{11} = a_1 f + b_1 f^1 + c_1 f^2 + d_1 f^3, \tag{1}$$

$$f^{13} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3 + e_2 f^{12}. \tag{2}$$

It will be supposed, in this case, that the other two three-point tangents at an arbitrary point are distinct. The three-spread therefore lies in an $S_5$.

Most of the results of the present discussion of this case, however, hold for any three-spread which satisfies two equations which can be reduced to the form of equations (1) and (2), and which therefore contains a system $\infty^2$ of curves the tangents to which count twice as three-point tangents.

Since the cones $\sum_{h=1}^{3}\sum_{k=1}^{3} e_{hk}\, du_h\, du_k = 0$ and $\sum_{h=1}^{3}\sum_{k=1}^{3} g_{hk}\, du_h\, du_k = 0$, at an arbitrary point $P$ on the three-spread, touch along $du_2 = du_3 = 0$, the tangent determined by $du_2 = du_3 = 0$ also counts twice as a fundamental tangent in the birational correspondence (¶ 15) between the tangents at $P$. Hence, the tangent $S_3$ at a consecutive point $(u_1 + du_1,\ u_2,\ u_3)$ on $u_2 = \text{const.}$, $u_3 = \text{const.}$ intersects the tangent $S_3$ at $(u_1,\ u_2,\ u_3)$ in a plane which passes through $du_2 = du_3 = 0$ and the discrete fundamental line.

26. Suppose, first, that the plane of intersection of consecutive tangent $S_3$ along an arbitrary curve of the system $u_2 = \text{const.}$, $u_3 = \text{const.}$ is invariant. The curve $u_2 = \text{const.}$, $u_3 = \text{const.}$ lies in this plane and the three-spread is touched along the curve by the plane.

The three-spreads which are touched along non-rectilinear plane curves by a system $\infty^2$ of planes will be discussed in Case 5. Suppose therefore, for the present case, that the curves $u_2 = \text{const.}$, $u_3 = \text{const.}$ are straight lines. *The necessary and sufficient condition that a ruled three-spread be touched along each generator by a fixed plane is that the three-spread be generated by a system* $\infty^1$ *of developables or cones.*

That the condition is sufficient follows from the fact that the tangent plane to the developable or cone touches the three-spread along the generator. To show that the condition is necessary, let the equations of the three-spread be written in the form:

$$x_i = f_i(u_1,\ u_2,\ u_3) = g_i(u_2,\ u_3) + u_1 h_i(u_2,\ u_3), \qquad i = 0, 1, \ldots, n.$$

The tangent $S_3$ at $(u_1,\ u_2,\ u_3)$ is determined by the generator through the point and by the line $L$ joining $x_i = g_i^2 + u_1 h_i$ to $x_i = g_i^3 + u_1 h_i^3$. Since, for all values of $u_1$, the tangent $S_3$ contains a fixed plane, the line $L$ must, for all values of $u_1$, meet the plane in a fixed point or in a fixed line intersecting the generator through $(u_1,\ u_2,\ u_3)$. In either case, there exists a value of $u_1$ such that

$$a\,(g^2 + u_1 h^2) + b\,(g^3 + u_1 h^3) + c\,g + d\,h = 0.$$

By a suitable transformation of the parameters this equation may be reduced

either to $g^3 = h$ or else to $g^3 = 0$. Hence, the three-spread is generated either by developables or by cones.

27. Suppose, next, that the tangent $S_3$ along $u_2 = \text{const.}$, $u_3 = \text{const.}$ all contain a fixed straight line. Since three consecutive $S_3$ of the system all pass through each point of the curve $u_2 = \text{const.}$, $u_3 = \text{const.}$, the straight line common to the tangent $S_3$ must coincide with $u_2 = \text{const.}$, $u_3 = \text{const.}$ The three-spread is therefore ruled. Let its equations be:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 h_i(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

Equation (2) is now the condition that the six points

$$x_i = g_i, \; x_i = h_i, \; x_i = g_i^2, \; x_i = h_i^2, \; x_i = g_i^3, \; x_i = h_i^3, \quad i = 0, 1, \ldots, n, \tag{a}$$

lie in an $S_4$. Conversely, if these six points lie in an $S_4$, equation (2) is satisfied and the generators count twice as three-point tangents. But the six points (a) lie in $S_4$ if, and only if, the tangent planes to the two-spreads $x_i = g_i(u_2, u_3)$ and $x_i = h_i(u_2, u_3)$, at corresponding points, intersect in a point. Hence, *the necessary and sufficient condition that the rectilinear generators of a ruled three-spread count twice as three-point tangents to the three-spread is that they join corresponding points of two two-spreads the tangent planes to which, at corresponding points, intersect in a point.*

The tangent $S_3$ to the three-spread along the generator $u_2 = \text{const.}$, $u_3 = \text{const.}$ envelope a quadric hypersurface in the $S_4$ determined by the points (a). This hypersurface has the generator for double line. For, the tangent $S_3$ at $(u_1, u_2, u_3)$ is determined by the generator through it and the line joining

$$x_i = g_i^2 + u_1 h_i^2 \text{ to } x_i = g_i^3 + u_1 h_i^3, \qquad i = 0, 1, \ldots, n.$$

When $u_1$ varies, the latter line describes a regulus and the $S_3$ determined by it and the fixed generator envelopes a quadric hypersurface.

28. Suppose, finally, that the tangent $S_3$ along an arbitrary curve of the system $u_2 = \text{const.}$, $u_3 = \text{const.}$ have, at most, a point in common. We have seen that the tangent $S_3$ at $(u_1 + du_1, u_2, u_3)$ intersects the tangent $S_3$ at $(u_1, u_2, u_3)$ in a plane through $du_2 = du_3 = 0$ and the discrete fundamental line. Three consecutive tangent $S_3$ along $u_2 = \text{const.}$, $u_3 = \text{const.}$, therefore, intersect in a line. Since the osculating plane to $u_2 = \text{const.}$, $u_3 = \text{const.}$ lies in the

tangent $S_3$, this line goes through $(u_1, u_2, u_3)$. It coincides, in fact, with the discrete fundamental line at $(u_1, u_2, u_3)$. For, let

$$f^{22} = a_{22} f + b_{22} f^1 + c_{22} f^2 + d_{22} f^3 + e_{22} f^{12} + g_{22} f^{33}, \tag{3}$$

$$f^{23} = a_{23} f + b_{23} f^1 + c_{23} f^2 + d_{23} f^3 + e_{23} f^{12} + g_{23} f^{33} \tag{4}$$

be the remaining two differential equations satisfied by the three-spread. Then the line of intersection of three consecutive tangent $S_3$ passes through $(u_1, u_2, u_3)$ in the direction

$$\frac{e_2 \cdot e_2 \cdot g_{22} - 2\, e_2 g_{23} + 1}{e_2 \cdot e_2 \cdot (e_{23} g_{22} - e_{22} g_{23}) + e_2 e_{22} - e_{23}}\, du_1 = -\frac{du_2}{e_2} = du_3 .$$

But this is also the direction of the discrete fundamental line at $(u_1, u_2, u_3)$. Along each curve $u_2 = \text{const.}$, $u_3 = \text{const.}$, therefore, the discrete fundamental lines generate a developable or cone. This developable or cone, obviously, touches the three-spread along $u_2 = \text{const.}$, $u_3 = \text{const.}$ If the locus of the discrete fundamental lines is a developable, the osculating $S_3$ to the edge of regression to any point is the tangent $S_3$ to the three-spread at the corresponding point.

Conversely, let $u_2 = \text{const.}$, $u_3 = \text{const.}$ be a system of curves on a three-spread

$$x_i = f_i(u_1, u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

such that the tangent $S_3$ along an arbitrary curve of the system envelope a developable which touches the three-spread along the corresponding curve $u_2 = \text{const.}$, $u_3 = \text{const.}$ Let the equations of the three-spread generated by the edges of regression of these developables be

$$x_i = F_i(u_1, u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

It then follows that:

$$f_i = \alpha F_i + \beta F_i^1, \qquad f_i^2 = \alpha_2 F_i + \beta_2 F_i^1 + \gamma_2 F_i^{11} + \delta_2 F_i^{111}, \quad i = 0, 1, \ldots, n,$$

$$f_i^1 = \alpha_1 F_i + \beta_1 F_i^1 + \gamma_1 F_i^{11}, \quad f_i^3 = \alpha_3 F_i + \beta_3 F_i^1 + \gamma_3 F_i^{11} + \delta_3 F_i^{111}, \quad i = 0, 1, \ldots, n.$$

From these equations, it follows that the three-spread $x_i = f_i$ satisfies equations (1) and (2). It is similarly seen that if the tangent $S_3$ to the given three-spread along $u_2 = \text{const.}$, $u_3 = \text{const.}$ envelope a cone which touches the three-spread along $u_2 = \text{const.}$, $u_3 = \text{const.}$, then equations (1) and (2) are satisfied. Hence, *the necessary and sufficient condition that a system $\infty^2$ of twisted curves which generate a three-spread count twice as three-point tangent curves, is that the tangent $S_3$ to the three-spread along each curve of the system envelope a developable or cone which touches the three-spread along the curve corresponding to it.*

*Case 3. Two Pairs of Three-Point Tangents Are Consecutive.*

29. Let the directions of the two three-point tangents at an arbitrary point be $du_2 = du_3 = 0$ and $du_1 = 0$, $du_3 = a\, du_2$. Denote $\dfrac{\partial}{\partial u_2} + a \dfrac{\partial}{\partial u_3}$ by $\Delta$. Since the three-spread has only a finite number of three-point tangents, $\Delta f^1$ does not lie in the tangent $S_3$. The four differential equations may therefore be reduced to:

$$f^{11} = a_1 f + b_1 f^1 + c_1 \Delta f + d_1 f^3, \tag{1}$$
$$f^{13} = a_2 f + b_2 f^1 + c_2 \Delta f + d_2 f^3 + e_2 \Delta f^1, \tag{2}$$
$$\Delta(\Delta f) = a_3 f + b_3 f^1 + c_3 \Delta f + d_3 f^3, \tag{3}$$
$$\Delta f^3 = a_4 f + b_4 f^1 + c_4 \Delta f + d_4 f^3 + e_4 \Delta f^1. \tag{4}$$

From (1), the point $x_i = \Delta f^{11}$ is seen to lie in the $S_4$ determined by the five points:

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \quad x_i = \Delta f_i^1, \qquad i = 0, 1, \ldots, n. \tag{5}$$

Hence the point $x_i = \dfrac{\partial}{\partial u_1}(\Delta f_i^1) = \Delta f_i^{11} + \dfrac{\partial a}{\partial u_1} f_i^{13}$ lies in this $S_4$. Similarly, it is seen from (3) that the point $x_i = \Delta(\Delta f_i^1)$ lies in the same $S_4$. Differentiating (2) with respect to $u_1$, $x_i = f_i^{113}$ is seen to lie in the $S_4$ of (5). Hence, differentiating (1) with respect to $u_3$, we find $d_1 = 0$, since $x_i = f_i^{33}$ can not lie in this $S_4$. In the corresponding way it can be shown that $d_3 = 0$. From (2) we find that $x_i = \Delta f^{13}$ lies in the $S_4$ of (5). Similarly, from (4), $x_i = \dfrac{\partial}{\partial u_1}(\Delta f^3)$ lies in this $S_4$. But $\dfrac{\partial}{\partial u_1}(\Delta f^3) = \Delta f^{13} + \dfrac{\partial a}{\partial u_1} f^{33}$. Hence, $\dfrac{\partial a}{\partial u_1} = 0$, since $x_i = f_i^{33}$ does not lie in the $S_4$ of (5). By a transformation of curvilinear coordinates of the form $u_1 = \bar{u}_1$, $u_2 = \phi(\bar{u}_2, \bar{u}_3)$, $u_3 = \psi(\bar{u}_2, \bar{u}_3)$, therefore, $a$ may be reduced to zero.

Suppose this transformation effected. The four equations now become:

$$f^{11} = a_1 f + b_1 f^1 + c_1 f^2, \tag{1'}$$
$$f^{13} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3 + e_2 f^{12}, \tag{2'}$$
$$f^{22} = a_3 f + b_3 f^1 + c_3 f^2, \tag{3'}$$
$$f^{23} = a_4 f + b_4 f^1 + c_4 f^2 + d_4 f^3 + e_4 f^{12}. \tag{4'}$$

From the first and third equations it follows that the two-spread $u_3 =$ const. has two distinct systems of three-point tangents at each point on it. It therefore lies in an $S_3$. This $S_3$ is determined by

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^{12}, \qquad i = 0, 1, \ldots, n.$$

The consecutive $S_3$ is determined, to infinitesimals of the second order, by

$$x_i=f_i+f_i^3du_3,\ x_i=f_i^1+f_i^{13}du_3,\ x_i=f_i^2+f_i^{23}du_3,\ x_i=f_i^{12}+f_i^{123}du_3,\ i=0,1,\ldots,n.$$

But $x_i=f_i^{23}$ and $x_i=f_i^{123}$ lie in the $S_4$ of

$$x_i=f_i,\quad x_i=f_i^1,\quad x_i=f_i^2,\quad x_i=f_i^3,\quad x_i=f_i^{12},\qquad i+0,1,\ldots,n.$$

Hence, the $S_3$ which contain any two consecutive two-spreads $u_3=$ const. lie in an $S_4$ and therefore intersect in a plane.

The equations of the three-spread can therefore always be put in one of the following forms:

$$\begin{aligned} x_i=f_i(u_1,u_2,u_3)&=l_0(u_1,u_2,u_3)\,g_i(u_3)+l_1(u_1,u_2,u_3)\,g_i^3(u_3)\\ &+l_2(u_1,u_2,u_3)\,g_i^{33}(u_3)+l_3(u_1,u_2,u_3)\,g_i^{333}(u_3),\quad i=0,1,\ldots,n;\end{aligned}\tag{6}$$

$$\begin{aligned} x_i=f_i(u_1,u_2,u_3)&=l_0(u_1,u_2,u_3)\,g_i(u_3)+l_1(u_1,u_2,u_3)\,g_i^3(u_3)\\ &+l_2(u_1,u_2,u_3)\,g_i^{33}(u_3)+l_3(u_1,u_2,u_3)\,M_i,\qquad i=0,1,\ldots,n;\end{aligned}\tag{7}$$

$$\begin{aligned} x_i=f_i(u_1,u_2,u_3)&=l_0(u_1,u_2,u_3)\,g_i(u_3)+l_1(u_1,u_2,u_3)\,g_i^3(u_3)\\ &+l_2(u_1,u_2,u_3)\,L_i+l_3(u_1,u_2,u_3)\,M_i,\qquad i=0,1,\ldots,n;\end{aligned}\tag{8}$$

$$\begin{aligned} x_i=f_i(u_1,u_2,u_3)&=l_0(u_1,u_2,u_3)\,g_i(u_3)+l_1(u_1,u_2,u_3)\,K_i\\ &+l_2(u_1,u_2,u_3)\,L_i+l_3(u_1,u_2,u_3)\,M_i,\qquad i=0,1,\ldots,n,\end{aligned}\tag{9}$$

where $K_i$, $L_i$ and $M_i$ are constants.

30. Conversely, any three-spread, the equations of which are of any one of these four forms, and on which the asymptotic lines of the three-dimensional surfaces $u_3=$ const. of the system

$$x_j=l_j(u_1,u_2,u_3),\qquad j=0,1,2,3,$$

are distinct, has, at an arbitrary point, two tangents each of which counts twice as a three-point tangent.

For, transform the parameters $u_1$ and $u_2$ in such a way that $u_2=$ const. and $u_3=$ const. are the asymptotic lines of the surfaces $u_3=$ const. Then

$$\begin{aligned} l_j^{11}&=a_1l_j+b_1l_j^1+c_1l_j^2,\qquad j=0,1,2,3;\\ l_j^{22}&=a_2l_j+b_2l_j^1+c_2l_j^2,\qquad j=0,1,2,3.\end{aligned}$$

From these equations, equations (1′) and (3′) follow.

Moreover, $x_i=f_i^{13}$ and $x_i=f_i^{23}$ lie in the $S_4$ determined by the tangent $S_3$ and $x_i=f_i^{12}$, from which equations (2′) and (4′) follow.

Hence, *the necessary and sufficient condition that a three-spread have, at an arbitrary point, two pairs of consecutive three-point tangents is that it be generated by a system* $\infty^1$ *of surfaces, the asymptotic lines of which are distinct and which lie in a system* $\infty^1$ *of* $S_3$ *such that consecutive* $S_3$ *intersect in a plane.* Such a three-spread does not, in general, lie in an $S_5$.

*Case* 4. *Three Three-Point Tangents Are Consecutive.*

31. Suppose, first, that the threefold three-point tangent curves are straight lines. Let the equations of the three-spread be:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 h_i(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

The conditions that the lines $u_2 = \text{const.}$, $u_3 = \text{const.}$ count three times as three-point tangent lines are that the three-spread satisfy the three equations:

$$f^{11} = 0, \tag{1}$$

$$f^{13} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3 + e_2 f^{12}, \tag{2}$$

$$f^{33} - 2e_2 f^{23} + e_2 \cdot e_2 f^{22} = a_3 f + b_3 f^1 + c_3 f^2 + d_3 f^3 + e_3 f^{12}. \tag{3}$$

By differentiating (2) with respect to $u_1$, we find that $d_2 e_2 = -\left(c_2 + \dfrac{\partial e_2}{\partial u_1}\right)$. But, by differentiating (2) with respect to $u_2$ and $u_3$ and (3) with respect to $u_1$, we find that $d_2 e_2 = 2\dfrac{\partial e_2}{\partial u_1} - c_2$. Hence $\dfrac{\partial e_2}{\partial u_1} = 0$. Hence, $e_2$ is independent of $u_1$. By a transformation of the parameters $u_2$ and $u_3$ which is independent of $u_1$, therefore, $e_2$ may be reduced to zero. Suppose this done. Since $c_2$ is now also zero, the two-spreads $u_2 = \text{const.}$ are either developables or cones.

32. First, let the two-spreads $u_2 = \text{const.}$ be developables. The equations of the three-spread may now be written in the form:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 g_i^3(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

Equations (1) and (2) are now satisfied. The $S_4$ of

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \quad x_i = f_i^{12}, \qquad i = 0, 1, \ldots, n,$$

coincides, for all values of $u_1$, with the $S_4$ of

$$x_i = g_i, \quad x_i = g_i^2, \quad x_i = g_i^3, \quad x_i = g_i^{23}, \quad x_i = g_i^{33}, \qquad i = 0, 1, \ldots, n.$$

Since $e_2 = 0$, (3) reduces to the condition that

$$g^{333} = a_4 g + b_4 g^2 + c_4 g^3 + d_4 g^{23} + e_4 g^{33}. \tag{4}$$

Hence, at any point $P$ of any curve $u_2 = \text{const.}$ on the two-spread $x_i = g_i(u_2, u_3)$, the osculating $S_3$ to the curve lies in the $S_4$ determined by the tangent planes to $x_i = g_i(u_2, u_3)$ at $P$ and at the consecutive point on $u_2 = \text{const.}$

Conversely, let

$$x_i = g_i(u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

be a two-spread in $S_n$ which contains a system $\infty^1$ of curves such that, at an arbitrary point $P$ of a curve of the system, the osculating $S_3$ to the curve lies in the $S_4$ determined by the tangent planes to the two-spread at $P$ and at the consecutive point along the curve. Such a two-spread satisfies a differential equation which can be reduced to the form (4). The tangents to the curves of the system generate a three-spread the equations of which then become:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 g_i^3(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

This three-spread satisfies equations (1), (2) and (3), and therefore its generators count thrice as three-point tangents.

33. Suppose, now, that the two-spreads $u_2 = \text{const.}$ are cones. Let the equations of the three-spread be written in the form:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 h_i(u_2), \qquad i = 0, 1, \ldots, n.$$

Since $e_2 = 0$, the condition that (3) be satisfied reduces to

$$g^{33} = a_5 g + b_5 g^2 + c_5 g^3 + d_5 h + e_5 h^2. \tag{5}$$

Hence, at any point $P$ of an arbitrary curve $u_2 = \text{const.}$ on $x_i = g_i(u_2, u_3)$, the osculating plane to the curve lies in the $S_4$ determined by the tangent plane to $x_i = g_i(u_2, u_3)$ at $P$ and the tangent line to $x_i = h_i(u_2)$ at the vertex of the cone through $P$.

Conversely, let $V$ be a two-spread and $C$ a curve in $S_n$. Let there correspond to the points $P_1$ of $C$ the curves $C_1$ of a system on $V$ such that the osculating plane to an arbitrary curve $C_1$ of the system at any point $P$ on it lies in the $S_4$ determined by the tangent plane to $V$ at $P_1$ and the tangent line to $C$ at $P_1$. Then the parametric equations $x_i = g_i(u_2, u_3)$ of $V$ and $x_i = h_i(u_2)$ of $C$ can be set up in such a way that (5) is satisfied for all values of $i$, and therefore that the three-spread

$$x_i = g_i(u_2, u_3) + u_1 h_i(u_2), \qquad i = 0, 1, \ldots, n,$$

generated by the lines joining the points of each curve $C_1$ to the corresponding point $P$ of $C$, satisfies (1), (2) and (3), and therefore has its generators for three-fold three-point tangents.

Hence, *the necessary and sufficient conditions that the rectilinear generators of a ruled three-spread count thrice as three-point tangents are* (*a*) *that the three-spread be generated by developables whose edges of regression form a system of curves $C$ on a two-spread $V$ such that the tangent $S_3$ to a curve $C$ of the system at a point $P$ on it lies in the $S_4$ determined by the tangent planes to $V$ at $P$ and at the consecutive point of $C$, or* (*b*) *that it be generated by cones projecting from the points of a curve $C$ a system of curves on a two-spread such that, at any point $P$ on the two-spread, the osculating plane to the curve through $P$ lies in the $S_4$ determined by the tangent plane to the two-spread at $P$ and the tangent to $C$ at the vertex of the cone which has the curve through $P$ for directrix.*

34. If, in addition to (1), (2) and (3), the three-spread satisfies a fourth homogeneous linear partial differential equation of the second order, it has, at each point, a fourth three-point tangent. This three-point tangent is, in general, distinct from the rectilinear generator, in which case the three-spread lies in an $S_5$. Conversely, if the three-spread lies in an $S_5$ it has, at each point, a fourth three-point tangent.

35. We shall now consider the case where the threefold three-point tangent curves $u_2 = \text{const.}$, $u_3 = \text{const.}$ are not straight lines. The conditions that the tangents to these curves count thrice as three-point tangents are:

$$f^{11} = a_1 f + b_1 f^1 + c_1 f^2 + d_1 f^3, \tag{1}$$

$$f^{13} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3 + e_2 f^{12}, \tag{2}$$

$$f^{33} - 2 e_2 f^{23} + e_2 e_2 f^{22} = a_3 f + b_3 f^1 + c_3 f^2 + d_3 f^3 + e_3 f^{13}. \tag{3}$$

Differentiating (1) with respect to $u_2$ and $u_3$ and with respect to $u_1$, we find:

$$d_1 (f^{33} - e_2 f^{23}) + c_1 (f^{23} - e_2 f^{22}) = a_4 f + b_4 f^1 + c_4 f^2 + d_4 f^3 + e_4 f^{12}.$$

Hence $e_2 = -\dfrac{c_1}{d_1}$, since, otherwise, the curves $u_2 = \text{const.}$, $u_3 = \text{const.}$ would count four times as three-point tangents. Hence, differentiating (1) with respect to $u_1$, we have:

$$f^{111} = a_5 f + b_5 f^1 + c_5 f^2 + d_5 f^3.$$

Hence, the curves $u_2 = \text{const.}$, $u_3 = \text{const.}$ are four-point tangents to the three-spread.

Conversely, let $u_2 = \text{const.}$, $u_3 = \text{const.}$ be a system of non-rectilinear four-point tangents to a three-spread:

$$x_i = f_i (u_1, u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

Equation (1) is at once satisfied. Moreover, $c_1$ and $d_1$ are not both zero, since $u_2 = \text{const.}$, $u_3 = \text{const.}$ are not straight lines. Let $d_1 \neq 0$. Differentiating (1) with respect to $u_1$, we obtain, since $x_i = f_i^{111}$ lies in the tangent $S_3$:

$$d_1 f^{13} = a_6 f + b_6 f^1 + c_6 f^2 + d_6 f^3 - c_1 f^{12}. \tag{2'}$$

Hence (2) is satisfied. Differentiating (1) with respect to $u_2$ and $u_3$ and (2′) with respect to $u_1$, we obtain:

$$d_1 d_1 f^{33} + 2 d_1 c_1 f^{23} + c_1 c_1 f^{22} = a_7 f + b_7 f^1 + c_7 f^2 + d_7 f^3 + e_7 f^{12}.$$

Hence (3) is satisfied and the curves $u_2 = \text{const.}$, $u_3 = \text{const.}$ are threefold three-point tangent curves.

Hence, *the necessary and sufficient condition that a system of non-rectilinear three-point tangent curves on a three-spread count thrice as three-point tangent curves is that the tangents to these curves be four-point tangents to the three-spread.*

If the three-spread satisfies, in addition to (1), (2) and (3), a fourth differential equation of the second order, it has at each point a fourth three-point tangent. This three-point tangent is, in general, distinct from the four-point tangent, in which case the three-spread lies in an $S_5$. Conversely, if the three-spread lies in an $S_5$, it has, at each point, a fourth three-point tangent which is, in general, distinct from the four-point tangent.

### *Case* 5. *Four Three-Point Tangents Are Consecutive.*

36. Suppose, first, that the fourfold three-point tangent curves $u_2 = \text{const.}$, $u_3 = \text{cont.}$ are straight lines. Let the equations of the three-spread be:

$$x_i = f_i(u_1, u_2, u_3) = g_i(u_2, u_3) + u_1 h_i(u_2, u_3), \qquad i = 0, 1, \ldots, n.$$

The conditions that the generators count four times as three-point tangents are:

$$f^{11} = 0, \tag{1}$$
$$f^{13} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3 + e_2 f^{12}, \tag{2}$$
$$f^{33} - e_2 f^{23} = a_3 f + b_3 f^1 + c_3 f^2 + d_3 f^3 + e_3 f^{12}, \tag{3}$$
$$f^{23} - e_2 f^{22} = a_4 f + b_4 f^1 + c_4 f^2 + d_4 f^3 + e_4 f^{12}. \tag{4}$$

Differentiating (2) with respect to $u_1$ and substituting $f_i = g_i + u_1 h_i$, we find that $e_2 = \dfrac{a u_1 + b}{c u_1 + d}$, where $a$, $b$, $c$ and $d$ are independent of $u_1$. But $ad - bc = 0$. For, otherwise, it follows from (3) and (4) that the three-spread lies in an $S_4$.

Hence $e_2$ is independent of $u_1$, and, by a transformation of the parameters $u_2$ and $u_3$, may be reduced to zero. The four equations now become:

$$f^{11} = 0, \tag{1'}$$
$$f^{13} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3, \tag{2'}$$
$$f^{33} = a_3 f + b_3 f^1 + c_3 f^2 + d_3 f^3 + e_3 f^{12}, \tag{3'}$$
$$f^{23} = a_4 f + b_4 f^1 + c_4 f^2 + d_4 f^3 + e_4 f^{12}. \tag{4'}$$

Differentiating (2′) with respect to $u_2$ and (4′) with respect to $u_1$, we find $c_2 = 0$. Hence, the two-spreads $u_2 = \text{const.}$ are developables or cones. Differentiating (1′), (2′) and (3′) successively with respect to $u_1$ and $u_3$, we find that the developable or cone $u_2 = \text{const.}$ lies in the $S_4$ determined by

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \quad x_i = f_i^{12}, \qquad i = 0, 1, \ldots, n.$$

But $x_i = f_i^2$, and its consecutive derivative points with respect to $u_1$ and $u_3$, also lie in this $S_4$. Hence, the consecutive two-spread, corresponding to the value $u_2 + du_2$ of the parameter, also lies in this $S_4$. Each two-spread, therefore, lies in the intersection of two consecutive $S_4$. Hence the three-spread is generated by a system $\infty^1$ of developables or cones lying in a system $\infty^1$ of $S_3$ such that consecutive $S_3$ lie in an $S_4$ and therefore intersect in a plane.

Conversely, it is seen, by an argument analogous to that given in Case 3, that a three-spread generated in this manner satisfies four equations of the type (1), (2), (3), (4) and, therefore, that its rectilinear generators count four times as three-point tangents.

Hence, *the necessary and sufficient condition that the generators of a ruled three-spread count four times as three-point tangent curves is that the three-spread be generated by a system $\infty^1$ of developables or cones lying in a system of $S_3$ such that consecutive $S_3$ intersect in a plane.* The three-spread does not, in general, lie in an $S_5$.

37. Suppose, now, that the fourfold three-point tangent curves $u_2 = \text{const.}$, $u_3 = \text{const.}$ are not straight lines. The conditions that the curves $u_2 = \text{const.}$, $u_3 = \text{const.}$ count four times as three-point tangents are:

$$f^{11} = a_1 f + b_1 f^1 + c_1 f^2 + d_1 f^3, \tag{1}$$
$$f^{13} - e_2 f^{12} = a_2 f + b_2 f^1 + c_2 f^2 + d_2 f^3, \tag{2}$$
$$f^{33} - e_2 f^{23} = a_3 f + b_3 f^1 + c_3 f^2 + d_3 f^3 + e_3 f^{12}, \tag{3}$$
$$f^{23} - e_2 f^{22} = a_4 f + b_4 f^1 + c_4 f^2 + d_4 f^3 + e_4 f^{12}. \tag{4}$$

From (3) and (4) we obtain at once:

$$f^{33} - 2\, e_2 f^{23} + e_2\, e_2 f^{22} = a_5 f + b_5 f^1 + c_5 f^2 + d_5 f^3 + e_5 f^{12}. \tag{5}$$

Moreover, $e_5 \neq 0$, since $du_1 = 0$, $du_2 + e_2 du_3 = 0$ does not determine a three-point tangent.

Differentiating (2) with respect to $u_2$ and $u_3$, we find that

$$x_i = f_i^{133} - 2e_2 f^{123} + e_2 e_2 f^{122}$$

lies in the $S_4$ of

$$x_i = f_i, \quad x_i = f_i^1, \quad x_i = f_i^2, \quad x_i = f_i^3, \quad x_i = f_i^{12}, \qquad i = 0, 1, \ldots, n. \tag{6}$$

Hence, by differentiating (5) with respect to $u_1$, we find that $x_i = f_i^{112}$ lies in this $S_4$. Differentiating (1) with respect to $u_1$, we now find that $c_1 + d_1 e_2 = 0$. Hence (1) may be written in the form:

$$f^{11} = a_6 f + b_6 f^1 + d_6 (f^3 - e_2 f^2). \tag{1$'$}$$

Differentiating this equation successively with respect to $u_1$, it is found that the curve $u_2 = \text{const.}$, $u_3 = \text{const.}$ lies in the $S_4$ of (6). It follows that this curve must be a plane curve. For, otherwise, by expressing $f^2$ and $f^3$ in terms of $f, f^1, f^{11}$ and $f^{111}$, it would be seen that the entire three-spread lay in this $S_4$. Hence:

$$f^{111} = a_6 f + b_6 f^1 + c_6 f^{11}. \tag{7}$$

From equations (1) and (7) all the others follow. Hence, *if a system of non-rectilinear plane curves are three-point tangent curves to the three-spread, they are fourfold three-point tangent curves.*

38. Let the equations of the three-spread be put in the form

$$x_i = \phi_0(u_1, u_2, u_3)\, g_{0i}(u_2, u_3) + \phi_1(u_1, u_2, u_3)\, g_{1i}(u_2, u_3) + \phi_2(u_1, u_2, u_3)\, g_{2i}(u_2, u_3),$$
$$i = 0, 1, \ldots, n.$$

By a linear transformation of $\phi_0, \phi_1, \phi_2$, the two-spreads $x_i = g_{0i}$, $x_i = g_{1i}$ and $x_i = g_{2i}$ may be taken to be two-spreads which lie on the three-spread. Suppose this transformation effected. For each pair of values of $u_2$ and $u_3$, the curve $u_2 = \text{const.}$, $u_3 = \text{const.}$ goes through the points $x_i = g_{0i}$, $x_i = g_{1i}$ and $x_i = g_{2i}$.

Equation (1$'$) now becomes:

$$\phi_0 (g_0^3 - e_2 g_0^2) + \phi_1 (g_1^3 - e_2 g_1^2) + \phi_2 (g_2^3 - e_2 g_1^2) = \alpha g_0 + \beta g_1 + \gamma g_2. \tag{1$''$}$$

Let $e_{20}$, $e_{21}$ and $e_{22}$ be the values of $e_2$ at $x_i = g_{0i}$, $x_i = g_{1i}$ and $x_i = g_{2i}$ respectively. Substituting into (1$''$) the values of $u_1$ which give rise to these three points and simplifying (1$''$) by means of the resulting equations, we obtain:

$$\phi_0 (e_{20} - e_2)\, g_0^2 + \phi_1 (e_{21} - e_2)\, g_1^2 + \phi_2 (e_{22} - e_2)\, g_2^2 = \alpha_1 g_0 + \beta_1 g_1 + \gamma_1 g_2.$$

But the plane of $x_i = g^2_{0i}$, $x_i = g^2_{1i}$, $x_i = g^2_{2i}$ can not have more than one point in the plane of $x_i = g_{0i}$, $x_i = g_{1i}$, $x_i = g_{2i}$. For, otherwise, the five points (6) would lie in an $S_3$ and the three-spread would satisfy five homogeneous linear partial differential equations of the second order. Hence,

$$\frac{\phi_0 (e_{20} - e_2)}{\sigma_0} = \frac{\phi_1 (e_{21} - e_2)}{\sigma_1} = \frac{\phi_2 (e_{22} - e_2)}{\sigma_2},$$

where $\sigma_0$, $\sigma_1$ and $\sigma_2$ are independent of $u_1$.

Eliminating $e_2$, we now obtain:

$$\phi_0 \phi_1 \sigma_2 (e_{20} - e_{21}) + \phi_1 \phi_2 \sigma_0 (e_{21} - e_{22}) + \phi_2 \phi_0 \sigma_1 (e_{22} - e_{20}) = 0.$$

Since this equation is of second degree in $\phi_0$, $\phi_1$ and $\phi_2$, the curves $u_2 =$ const., $u_3 =$ const. are conics.

Consider the four-spread generated by the planes of these conics,

$$x_i = g_{0i}(u_2, u_3) + v_1 g_{1i}(u_2, u_3) + v_2 g_{2i}(u_2, u_3), \qquad i = 0, 1, \ldots, n,$$

where $v_1$, $v_2$, $u_2$, $u_3$ are the parameters. The tangent $S_4$ at any point is the $S_4$ of the points (6). It is therefore invariant over each plane $u_2 =$ const., $u_3 =$ const.

39. Conversely, let

$$x_i = g_{0i}(u_2, u_3) + v_1 g_{1i}(u_2, u_3) + v_2 g_{2i}(u_2, u_3), \qquad i = 0, 1, \ldots, n, \tag{8}$$

be a four-spread generated by planes in such a way that the tangent $S_4$ is invariant along each plane. Let $x_i = \theta_i$ and $x_i = \psi_i$ be two of the six points,

$$x_i = g^2_{0i},\ x_i = g^2_{1i},\ x_i = g^2_{2i},\ x_i = g^3_{0i},\ x_i = g^3_{1i},\ x_i = g^3_{2i}, \qquad i = 0, 1, \ldots, n,$$

which do not lie in the plane $u_2 =$ const., $u_3 =$ const. Let

$$\begin{aligned} g^2_0 &= \alpha_1 g_0 + \beta_1 g_1 + \gamma_1 g_2 + \sigma_0 \theta + \tau_0 \psi, \\ g^3_0 &= \alpha_2 g_0 + \beta_1 g_1 + \gamma_1 g_2 + \pi_0 \theta + \rho_0 \psi, \end{aligned}$$

and similarly for $g^2_1$, $g^3_1$, $g^2_2$ and $g^3_2$.

Then the points in the plane $u_2 =$ const., $u_3 =$ const. which also lie in a consecutive plane must satisfy the equations:

$$\begin{aligned} du_2 (\sigma_0 + v_1 \sigma_1 + v_2 \sigma_2) + du_3 (\pi_0 + v_1 \pi_1 + v_2 \pi_2) &= 0, \\ du_2 (\tau_0 + v_1 \tau_1 + v_2 \tau_2) + du_3 (\rho_0 + v_1 \rho_1 + v_2 \rho_2) &= 0. \end{aligned}$$

They therefore lie on the conic:

$$\begin{vmatrix} \sigma_0 + v_1 \sigma_1 + v_2 \sigma_2 & \pi_0 + v_1 \pi_1 + v_2 \pi_2 \\ \tau_0 + v_1 \tau_1 + v_1 \tau_2 & \rho_0 + v_1 \rho_1 + v_2 \rho_2 \end{vmatrix} = 0. \tag{9}$$

This conic may be composite. Its locus is then a ruled three-spread. Excluding this case, let

$$v_1 = \frac{\phi_1(u_1, u_2, u_3)}{\phi_0(u_1, u_2, u_3)}, \qquad v_2 = \frac{\phi_2(u_1, u_2, u_3)}{\phi_0(u_1, u_2, u_3)}$$

satisfy (9) identically. Then the three-spread,

$$\begin{aligned} x_i = f_i(u_1, u_2, u_3) = \phi_0(u_1, u_2, u_3)\, g_{0i}(u_2, u_3) + \phi_1(u_1, u_2, u_3)\, g_{1i}(u_2, u_3) \\ + \phi_2(u_1, u_2, u_3)\, g_{2i}(u_1, u_2), \qquad i = 0, 1, \ldots, n, \end{aligned} \tag{10}$$

satisfies two equations of the forms (1′) and (7). The conics $u_2 = \text{const.}$, $u_3 = \text{const.}$ on it therefore count four times as three-point tangent curves.

Since the conics (9) are not composite, the four-spread (8) satisfies nine homogeneous linear partial differential equations of the second order. Since the four-spread is not generated by $S_3$ in such a way that consecutive $S_3$ intersect in a plane, it lies in an $S_5$. Hence, the three-spread (10) lies in an $S_5$.

Hence, *the necessary and sufficient condition that a system $\infty^2$ of non-rectilinear plane curves which generate a three-spread be three-point tangent curves to the three-spread, is that they be the system of conics in which each plane of a four-spread which is generated by planes in such a way that the tangent $S_4$ is invariant along each plane is intersected by the planes consecutive to it. The conics then count four times as three-point tangent curves to the three-spread. The three-spread generated by such a system of conics lies in an $S_5$.*

The three-spread is, in this case, touched along the conics by the planes of the conics. When a three-spread generated by a system $\infty^2$ of plane curves is touched along the curves by the planes of the curves, the tangents to the curves are three-point tangents to the three-spread. Hence:

*The necessary and sufficient condition that a three-spread be touched along non-rectilinear curves by a system $\infty^2$ of planes is that the curves of contact be conics and that the plane of each conic meet the planes consecutive to it in the points of the conic.*

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